

Asymptotic Bohr Radius for the Polynomials in One Complex Variable

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ABSTRACT. We consider the Bohr radius R_n for the class of complex polynomials in one variable of degree at most n . It was conjectured by R. Fournier in 2008 that $R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + o(\frac{1}{n^2})$. We shall prove this conjecture is true in this paper.

1. Introduction

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} and H^∞ be the Banach space of bounded analytic functions on \mathbb{D} with the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

Also let \mathcal{P}_n denote the subspace of H^∞ consisting of all the complex polynomials of degree at most n . The Bohr radius R for H^∞ is defined as

$$R = \sup\{r \in (0, 1) : \sum_{k=0}^{\infty} |a_k| r^k \leq \|f\|_\infty, \text{ for all } f(z) = \sum_{k=0}^{\infty} a_k z^k \in H^\infty\}.$$

Bohr's famous power series theorem [1] shows that $R = \frac{1}{3}$.

In 2004, Guadarrama [4] considered the Bohr type radius for the class \mathcal{P}_n defined by

$$(1.1) \quad R_n = \sup\{r \in (0, 1) : \sum_{k=0}^n |a_k| r^k \leq \|p\|_\infty, \text{ for all } p(z) = \sum_{k=0}^n a_k z^k \in \mathcal{P}_n\},$$

and gave the estimate

$$\frac{C_1}{3^{n/2}} < R_n - 1/3 < C_2 \frac{\log n}{n},$$

for some positive constants C_1 and C_2 . Later in 2008, Fournier obtained an explicit formula for R_n by using the notion of bounded preserving operators. He proved the following theorem [2]

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Theorem 1.1. For each $n \geq 1$, let $T_n(r)$ be the following $(n+1) \times (n+1)$ symmetric Toeplitz matrix

$$(1.2) \quad \begin{pmatrix} 1 & r & -r^2 & r^3 & \cdots & (-1)^{n-1}r^n \\ r & 1 & r & -r^2 & \cdots & (-1)^{n-2}r^{n-1} \\ -r^2 & r & 1 & r & & \\ r^3 & -r^2 & r & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{n-1}r^n & \cdots & & r & & 1 \end{pmatrix}.$$

Then R_n is equal to the smallest root in $(0, 1)$ of the equation

$$\det T_n(r) = 0.$$

Based on the numerical evidence, he conjectured that

$$R_n = \frac{1}{3} + \frac{\pi^2}{3n^2} + \dots$$

The purpose of this note is to provide a positive answer. We shall prove

Theorem 1.2. Let R_n be as in (1.1), then

$$\lim_{n \rightarrow \infty} n^2 \left(R_n - \frac{1}{3} \right) = \frac{\pi^2}{3}.$$

2. Main Theorem

In this section, we prove Theorem 1.2. The methods we use are similar to that in [3, Chapter 5].

Proof of Theorem 1.2. Let $\Delta_n = \Delta_n(r) = \det T_n(r)$, where $T_n(r)$ is the symmetric Toeplitz matrix (1.2). By Theorem 1.1, R_n is the smallest root in $(0, 1)$ of the equation

$$(2.1) \quad \Delta_n(r) = 0.$$

For $n \geq 2$, multiplying the second row of Δ_n by r , adding it to the first row and performing a similar operation with the columns, we have

$$\Delta_n(r) = \det \begin{pmatrix} 1+3r^2 & 2r & 0 & \cdots & 0 \\ 2r & 1 & r & -r^2 & \cdots & (-1)^{n-2}r^{n-1} \\ 0 & r & 1 & r & & \\ & -r^2 & r & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & \cdots & & r & & 1 \end{pmatrix}$$

$$(2.2) \quad = (3r^2 + 1)\Delta_{n-1}(r) - 4r^2\Delta_{n-2}(r).$$

If we set $\Delta_{-1}(r) = 1$, then the recurrence relation (2.2) holds for all $n \geq 1$.

Consider the function associated with these Toeplitz matrices Δ_n

$$f(r, \theta) = 1 + \sum_{|n|>0} (-1)^{n-1} r^n e^{in\theta} = \frac{3r^2 + 4r \cos \theta + 1}{r^2 + 2r \cos \theta + 1}.$$

In order to solve the equation (2.1), suppose

$$(2.3) \quad r = g(x) = \frac{1}{3}(-2 \cos x - \sqrt{4 \cos^2 x - 3}),$$

for some $x \in [0, \pi]$. This substitution comes from

$$3r^2 + 4r \cos x + 1 = 0 = f(r, x).$$

(In fact, for a fixed r , every eigenvalue λ of T_n can be written as $\lambda = f(r, x)$, for some $x \in [0, \pi]$. See [3, Chapter 5].)

Then (2.2) becomes

$$\Delta_n = (-4r \cos x) \Delta_{n-1} - 4r^2 \Delta_{n-2}.$$

Its characteristic equation

$$\lambda^2 + 4r \cos x \lambda + 4r^2 = 0$$

has the roots $-2re^{\pm ix}$. Adding the initial conditions $\Delta_{-1} = \Delta_0 = 1$, we have

$$\Delta_n = \frac{(-2r)^{n+1}}{1 - r^2} \left(\frac{\sin(n+2)x}{\sin x} + 2r \frac{\sin(n+1)x}{\sin x} + r^2 \frac{\sin nx}{\sin x} \right).$$

Denote

$$p_n(\cos x) = \frac{\sin(n+2)x}{\sin x} + 2r \frac{\sin(n+1)x}{\sin x} + r^2 \frac{\sin nx}{\sin x}.$$

Then $p_n(t)$ is a polynomial of degree $n+1$ in $t = \cos x$. Let

$$x_\nu = \frac{\nu\pi}{n+2}, \quad \nu = 1, 2, \dots, n+1.$$

Direct computation shows that

$$p_n(\cos x_\nu) = (-1)^\nu 2r(1 + r \cos \nu),$$

thus

$$\operatorname{sgn} p_n(\cos x_\nu) = (-1)^\nu.$$

Also

$$\lim_{x \rightarrow 0^+} p_n(\cos x) > 0.$$

So p_n has $n+1$ distinct zeros $\{\cos t_\nu^{(n)} | \nu = 1, 2, \dots, n+1\}$, such that

$$(2.4) \quad 0 < t_1^{(n)} < x_1 < t_2^{(n)} < x_2 < \dots < t_{n+1}^{(n)} < x_{n+1} < \pi.$$

That means every root of the equation (2.1) has the form (2.3). Notice that g is positive only on $[\frac{5\pi}{6}, \pi]$ and decreasing on $[\frac{5\pi}{6}, \pi]$, so the smallest root of (2.1) in the interval $(0, 1)$ is $g(t_{n+1}^{(n)})$, i.e. $R_n = g(t_{n+1}^{(n)})$.

Next, we will find an asymptotic expression for $t_{n+1}^{(n)}$. Notice that

$$(2.5) \quad \lim_{n \rightarrow \infty} \frac{p_n(\cos \frac{z}{n+2})}{n+2} = (1+r)^2 \frac{\sin z}{z}.$$

And (2.5) holds uniformly for $|z| < 2\pi$. Let

$$t_{n+1}^{(n)} = \frac{(n+1)\pi - \epsilon_n}{n+2},$$

then $\epsilon_n \in (0, \pi)$ by relation (2.4). Thus

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \frac{p_n(\cos \frac{(n+1)\pi - \epsilon_n}{n+2})}{n+2} \\ &= \lim_{n \rightarrow \infty} (1+r)^2 \frac{\sin((n+1)\pi - \epsilon_n)}{(n+1)\pi - \epsilon_n}. \end{aligned}$$

Consequently, $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and then

$$R_n = g(\pi - \theta_n),$$

where

$$\theta_n = \frac{\pi}{n} + o\left(\frac{1}{n}\right).$$

By (2.3),

$$\begin{aligned} r &= \frac{1}{3}(2 \cos \theta_n - \sqrt{4 \cos^2 \theta_n - 3}) \\ &= \frac{1}{3} + \frac{\pi^2}{3n^2} + o\left(\frac{1}{n^2}\right). \end{aligned}$$

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References

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